Theoretical Model of the Boundary Condition at a Fluid-Porous Interface

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An equation for the volume average fluid velocity in a boundary layer region between a fluid and a porous medium is derived. The result is valid for a general anisotropic porous medium subject to the restrictions imposed by the assumptions made in the analysis. The basic equations of motion for an incompressible Newtonian fluid with negligible inertia are used and a linear map M_{ij} from the volume average velocity $\langle v \rangle$ into the point velocity v is assumed to exist. Using scaling arguments, an equation for $\langle v \rangle$ is found which reduces to the Darcy equation for a point entirely within the porous medium below the boundary region and to the usual Navier-Stokes equation for a point entirely within the fluid above the boundary region. The form of the equation contains two of the terms that were originally hypothesized and tested experimentally by Beavers and Joseph (1967). In particular, the following equation is obtained:

$$\langle v_i \rangle = -\frac{K_{ij}}{\mu}\frac{\partial}{\partial x_j}\langle P \rangle + K_{ij}L_{jmk}\frac{\partial}{\partial x_k}\langle v_m \rangle + K_{ij}N_{jm}\nabla^2\langle v_m \rangle$$

where the tensors K_{ij}, L_{jmk} and N_{jm} are mathematically defined in terms of the map M_{ij}.

SCOPE

The analysis developed in this paper is an attempt to derive, from basic principles, a valid equation for the average velocity at the interface between an incompressible Newtonian fluid and a fixed porous medium. Since the familiar "no-slip" condition is clearly unsatisfactory for highly premeable boundaries, a slip boundary condition of some sort is required to solve the field equations for the motion in an exterior fluid region.

An empirical equation for tangential velocity at the interface between a Newtonian fluid and an isotropic porous medium was first suggested by Beavers and Joseph (1967). In our notation, this equation can be expressed as:

$$\langle v_1(0) \rangle = -\frac{K}{\mu}\frac{d}{dx}\langle P(-L) \rangle + \frac{K^{1/2}}{\alpha}\frac{d}{dy}\langle v_1(0) \rangle$$

where $\langle v_1 \rangle$ is the average velocity in the x direction, $\langle P \rangle$ is the average pressure, μ is the fluid viscosity, K is the permeability of the porous medium, and α is a dimensionless quantity that supposedly depends only on the structure of the porous medium. The parameter α was not defined mathematically and values of α were assumed to evaluate any data. The mean interface is at $y = 0$, and the first term represents the Darcy flow in the porous medium below the boundary region at $y \leq -L$.

Experimental verification of the boundary condition proposed was based on a comparison of the increase in mass flow rate predicted by this slip condition with respect to a no-slip condition. This approach was essentially followed in latter papers by Beavers et al. (1970, 1974) in which the form of the slip boundary condition was verified. The results, however, compared predicted and experimentally determined integrated values (of mass flow rate) and so the exact nature of the boundary condition was not considered. The value of the parameter α did appear to be independent of the type of fluid (1974); however, the results were not totally conclusive.

The above boundary condition was also verified experimentally by Taylor (1971) using a simple geometric model for the porous medium. The fluid flow using the slip condition was analyzed theoretically in a companion paper by Richardson (1971) to compare theoretical and experimental results. For this special geometry, the parameter α was a function of the distance between the fluid-porous interface and the upper solid boundary so that it was not dependent solely on the structure of the porous medium. The geometry of the model used was very specialized, however, and was not comparable to the isotropic porous blocks used in Beaver's tests.

A theoretical development of the slip boundary condition was published by Saffman (1971). A boundary layer type of analysis via matching a Darcy flow for $y < 0$ and a Navier-Stokes creep flow for $y > 0$ yielded the result:

$$\langle v_1 \rangle = \frac{K^{1/2}}{\alpha}\frac{d}{dy}\langle v_1 \rangle + 0(K) \quad \text{at } y = 0$$

In this analysis, α is actually a constant of integration of an asymptotic solution and so it is not precisely defined in terms of the structure of the porous medium. The boundary layer is assumed to be on the scale of the microscopic dimension d, and so since $K = 0(d^2)$, the $0(K)$ (e.g., Darcy equation) terms are neglected with respect to the larger $0(K^{1/2})$ terms.

The theory presented here is basically an extension of some of the work of the previous authors, notably Whitaker (1970), for application to the specific question at hand. The resulting boundary condition equation is valid for a more general (i.e., anisotropic) porous medium than formerly considered and the presence of an arbitrary dimensionless parameter is eliminated. For an isotropic medium, an equation for α is obtained in terms of the linear mapping M_{ij} from the average velocity to the point velocity. Furthermore, in addition to the two terms assumed by Beavers and Joseph, there is a third term proportional to $\nabla^2\langle v \rangle$ that must be included in the slip condition if all the same orders of magnitude terms are retained.

CONCLUSIONS AND SIGNIFICANCE

The analysis of the fluid flow in the "boundary" region between a Newtonian fluid and a porous medium developed in this paper has resulted in a "slip" boundary condition equation for the average fluid velocity in terms of gradients of average pressure and velocity and, indirectly, the geometry of the porous interface.

The model is based on the basic principles of fluid mechanics for a Newtonian fluid undergoing steady, incompressible, negligible inertia flow with constant viscosity within the boundary region.

Some of the conclusions resulting from the analysis are summarized below:

1. The general slip boundary condition which is valid at the mean boundary surface gives the average velocity in terms of gradients of average pressure and first and second derivatives of the average velocity. There are three terms present which are of the same order according to the scaling arguments used in the development of the model.

2. The first two terms of the derived boundary condition are of the same form as the empirical slip boundary condition proposed by Beavers et al. (1967, 1970, 1974). These terms consist of a Darcy flow term (i.e., proportional to the pressure gradient) and a first-order derivative term in velocity. According to this model, however, it is necessary to include a third term which is not present in previous theoretical (Saffman, 1971) or empirical slip boundary conditions.

3. For the special case of isotropic media, a previously undefined dimensionless parameter (α) present in Beavers et al. equation can be expressed in terms of a line integral and a surface integral of the linear map of the average velocity into the point velocity.

4. The slip boundary condition derived in this paper reduces to the basic negligible inertia viscous flow of an incompressible Newtonian fluid above the boundary region (i.e., the all fluid space) and to the usual Darcy equation for flow through a porous medium below the boundary region (i.e., the uniformly porous space).

PHYSICAL DESCRIPTION AND SCALING

Of interest in this problem is the region between the fluid space $y \geq L$ and the interior of the porous medium $y \leq -L$ where L is the scale through which the porosity $\Phi(x)$ changes from a (not necessarily constant) value representative of the porous medium as a whole to the value of 1 in the Newtonian fluid above the boundary (Figure 1).

The medium is described by a void volume distribution function β where

$$\beta(x) = \begin{cases} 1 \text{ if } x \epsilon V_f \text{ (fluid space)} \\ 0 \text{ if } x \epsilon V_s \text{ (solid space)} \end{cases} \tag{1.1}$$

Thus, $V_f = \int_V \beta dV$ is the total fluid volume within the averaging volume V and so $V_s = V - V_f$ is the total solid volume within V (Figure 2).

The volume V is chosen as the smallest volume above which all continuous tensor fields $\psi(x)$ (e.g., stress, density, velocity, etc.) defined in the fluid space are "smoothed out" when averaged over V (Whitaker, 1970) (Figure 3). We let $\psi = 0$ for a point x in the solid space and so ψ is defined for all $x \epsilon V$; hence, the volume (phase) average can be written as:

$$\langle \psi(x) \rangle = \frac{1}{V} \int_V \psi dV. \tag{1.2}$$

In this analysis, V is taken to be a constant volume element of $0(l^3)$; since the porous structure is not necessarily uniform, $\langle \psi \rangle$ is a function of x in general. Clearly, we require that $l \gg d$ where d is a microscopic scale (e.g., pore diameter) for this slowly varying condition to be valid.

From the definition of β,

$$V_f(x) = \int_V \beta dV = V \langle \beta \rangle \tag{1.3}$$

where the porosity

$$\Phi(x) = \frac{V_f}{V} = \langle \beta \rangle. \tag{1.4}$$

Now consider the order of magnitude of gradients of the average

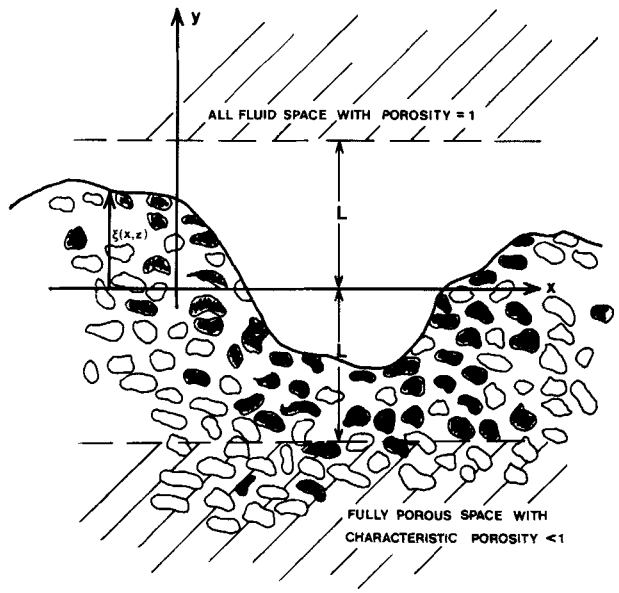

Figure 1. The boundary layer region is defined for $-L \leq y \leq L$ where the porosity Φ changes from 1 at $y = L$ to a smaller characteristic value for $y = -L$. The xz plane represents the mean surface having equal volume above and below the plane so that $\int \zeta dxdz = 0$

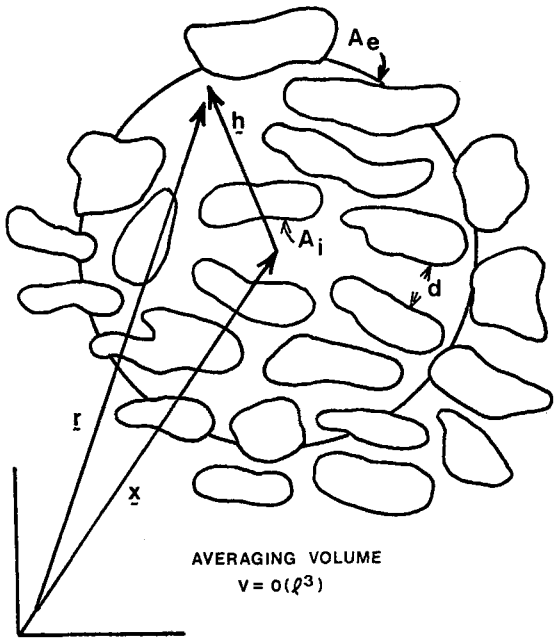

Figure 2. Averaging volume V of $0(\ell^3)$ for which all averaged continuous tensor fields are slowly varying (i.e., "smooth"). The solid space V_s and the fluid space V_f constitute the total volume. Area A_e represents the entrance and exit fluid areas of V and the solid-fluid boundaries within V are denoted by A_i. The dimension d is a microscopic scale that characterizes the grain or pore size of the medium. x is considered to be a fixed point at the centroid of V and h varies over the volume. Since $V = 0(\ell^3)$, then $h = 0(\ell)$.

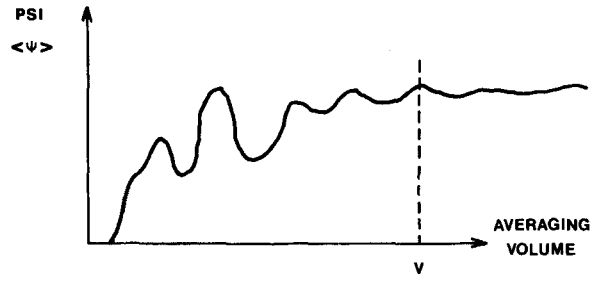

Figure 3. The averaged tensor field $\langle\psi\rangle$ becomes smooth for a volume V or larger.

velocity and the point velocity. Suppose that U is the characteristic velocity of the fluid above $y = L$ and $\hat{U}$ is the characteristic velocity given by the Darcy equation in the porous medium below $y = -L$ (Figure 4). Within the porous medium ($y \leq -L$), the point velocity as seen by x_4 and x_5 is scaled via $\partial v/dy = 0(\hat{U}/d)$ and the average velocity is scaled via $\partial\langle v\rangle/\partial y = 0(\hat{U}/L_p)$ if we refer to x_4 and a point on the solid wall. On the other hand, for points within the boundary layer region ($-L \leq y \leq L$), the average velocity is scaled according to $\partial\langle v\rangle/\partial y = 0(U/L)$ and the point velocity scale has two possibilities depending on the location of the points inside the boundary layer used for scaling purposes. If we scale the velocity gradient with respect to points x_2 and x_3 we have $\partial v/\partial y = 0(\hat{U}/d)$, whereas if we use points x_1 and x_2, the result is $0(U/L)$ (Figure 4). For this problem, the dominant scale with respect to the significance of terms in the basic equations is the latter, so that for $-L \leq y \leq L$,

$$\nabla v = 0\left(\frac{U}{L}\right), \tag{1.5}$$

and

$$\nabla\langle v\rangle = 0\left(\frac{U}{L}\right).$$

MAPPING

Both v and $\langle v\rangle$ are continuous real functions ϵR^3 and one can be mapped into the other. We assume that there exists a linear transformation $M_{ij}(x)$ that maps the average velocity into the point velocity via

$$v_i = M_{ij}\langle v_j\rangle \tag{2.1}$$

where repeated indices are summed. The existence of a linear transformation is valid within the porous medium $y \leq -L$ as shown by Whitaker (1970) and in the fluid region $y \geq L$ where $M_{ij} = \delta_{ij}$ since $v \cong \langle v\rangle$. Because of the nature of the scaling in this particular problem, it is not obvious that M_{ij} is linear; however, it seems reasonable to assume so.

The inverse map M_{ij}^{-1} exists for all x if and only if $v_i = M_{ij}\langle v_j\rangle = 0$ implies $\langle v_j\rangle = 0$ (Halmos, 1958). For $x\epsilon V_s$ the point velocity $v = 0$, but the average velocity $\langle v\rangle$ is not necessarily 0 so that M_{ij}^{-1} may not exist there.

The average of this map can be shown to be $\langle M_{ij}\rangle = \delta_{ij}$ by taking the average of both sides of Eq. 2.1. Using Eq. A2 we have $\langle v_i\rangle = \langle M_{ij}\rangle\langle v_j\rangle$ for $l \ll L$ which gives the desired result.

EQUATION FOR $\langle v\rangle$ IN THE BOUNDARY LAYER

The Navier-Stokes equations for steady, viscous, incompressible flow with negligible inertia are assumed to be valid within the fluid spaces between the solid particles or in the "pores," i.e.

$$0 = -\nabla P + \mu\nabla^2 v \tag{3.1}$$

at any point $x\,\epsilon V_f$. Now consider an arbitrary curve lying entirely within the fluid space having arc length s varying from 0 to s_b

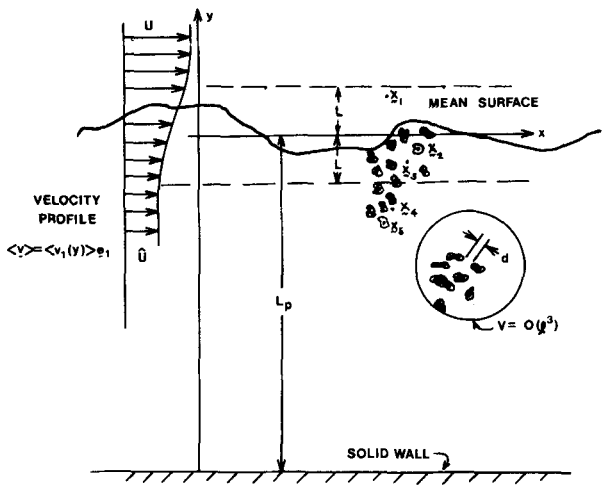

Figure 4. L_p is the characteristic length scale of the macroscopic process, U and $\hat{U}$ are the characteristic velocities in the fluid region $y \geq L$ and in the porous region $y \leq -L$, respectively. A rectilinear velocity profile $\langle v\rangle = \langle v_1(y)\rangle e_1$ is also shown.

(Figure 5). The unit tangent to the curve is $e_s = dx/ds$ so that $e_s\cdot\nabla = d/ds$ along the arc. Taking the scalar product of e_s with Eq. 3.1 gives

$$\frac{dP}{ds} = \mu e_{s_i}\nabla^2 v_i \tag{3.2}$$

and substituting Eq. 2.1 into the Laplacian of v_i yields

$$\frac{dP}{ds} = \mu e_{s_i}[\nabla^2 M_{ij}\langle v_j\rangle + 2\nabla M_{ij}\cdot\nabla\langle v_j\rangle + M_{ij}\nabla^2\langle v_j\rangle] \tag{3.3}$$

If we integrate along the curve from $s = 0$ to $s_b(x)$, the resulting equations are

$$P(x) - P(x_0) = \int_0^{s_b(x)} \frac{dp}{ds}\,ds,$$

and

$$P(x) - P(x_0) = \mu\int_0^{s_b} e_{s_i}\nabla^2 M_{ij}\langle v_j\rangle ds + 2\mu\int_0^{s_b} e_{s_i}\frac{\partial M_{ij}}{\partial x_k}\frac{\partial\langle v_j\rangle}{\partial x_k}\,ds + \mu\int_0^{s_b} e_{s_i}M_{ij}\nabla^2\langle v_j\rangle ds. \tag{3.4}$$

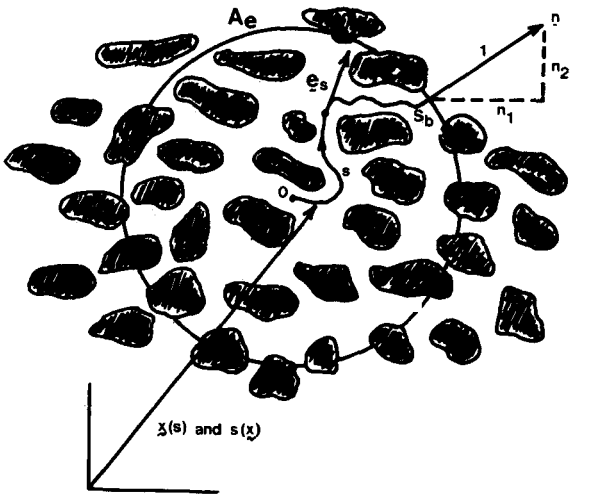

Figure 5. Arbitrary curve between any two points in the fluid space V_f. For mathematical convenience, it has been drawn between the centroid of V and a point on A_e. The arc length s varies from 0 to s_b and position x on the curve is a function of s. The unit tangent to the curve at x is denoted by $e_s = dx/ds$, and the unit outer normal to A_e is denoted by n.

Assuming $l \ll L$ and $s_b = 0(l)$, Eq. A5 can be applied to remove gradients of $\langle v_j \rangle$ from the line integrals so that

$$P(x) - P(x_0) = -\mu m_j \langle v_j \rangle + 2\mu \frac{\partial q_j}{\partial x_k} \frac{\partial \langle v_j \rangle}{\partial x_k} + \mu r_j \nabla^2 \langle v_j \rangle \quad (3.5)$$

where we define

$$m_j = -\int_0^{sb} e_{s_i} \nabla^2 M_{ij} ds, \quad (3.6)$$

$$\frac{\partial q_j}{\partial x_k} = \int_0^{sb} e_{s_i} \frac{\partial M_{ij}}{\partial x_k} ds, \quad (3.7)$$

and

$$r_j = \int_0^{sb} e_{s_i} M_{ij} ds. \quad (3.8)$$

Note that m_j, $\partial q_j / \partial x_k$ and r_j are functions of x and of the structure of the porous medium.

To express Eq. 3.5 in terms of averaged quantities, it is necessary to apply a relationship between the gradient of an integral over V_f to an area integral over the exit and entrance area A_e. This equation has been presented in previous literature (Slattery, 1967; Whitaker, 1970) and can be derived by considering the Jacobian of the transformation of $x(X,s)$ where $X = x(X,0)$ and s is the arc length of an arbitrary curve. That is,

$$\nabla \int_{V_f} \psi dV = \int_{A_e} \psi n dA. \quad (3.9)$$

If Eq. 4.9 is now applied with $\psi(x) = P(x) - P(x_0)$, we have

$$\nabla \left\{ \frac{1}{V} \int_{V_f} [P(x) - P(x_0)] dV \right\} = \frac{1}{V} \int_{A_e} [P(x) - P(x_0)] n dA. \quad (3.10)$$

Requiring $P(x) - P(x_0) = 0$ for $x \epsilon V_s$ allows us to replace the volume V_f by the total averaging volume V which is a constant so that

$$\nabla \{ \langle P(x) \rangle - P(x_0) \} = \frac{1}{V} \int_{A_e} [P(x) - P(x_0)] n dA. \quad (3.11)$$

Substituting Eq. 3.5 into 3.11 and noting that $\nabla P(x_0) = 0$ since $P(x_0)$ is a constant, we obtain

$$\nabla \langle P(x) \rangle = -\frac{\mu}{V} \int_{A_e} m_j \langle v_j \rangle n dA + \frac{2\mu}{V} \int_{A_e} \frac{\partial q_j}{\partial x_k} \frac{\partial \langle v_j \rangle n}{\partial x_k} dA + \frac{\mu}{V} \int_{A_e} r_j \nabla^2 \langle v_j \rangle n dA. \quad (3.12)$$

Derivatives of $\langle v_j \rangle$ of all orders can be removed from the integrals if $l \ll L$ (Appendix Eq. A4) so that for the ith component of the gradient of $\langle P \rangle$, we find

$$\frac{\partial}{\partial x_i} \langle P(x) \rangle = -\mu \left\{ \frac{1}{V} \int_{A_e} n_i m_j dA \right\} \langle v_j \rangle + \mu \left\{ \frac{2}{V} \int_{A_e} n_i \frac{\partial q_j}{\partial x_k} dA \right\} \frac{\partial \langle v_j \rangle}{\partial x_k} + \mu \left\{ \frac{1}{V} \int_{A_e} n_i r_j dA \right\} \nabla^2 \langle v_j \rangle. \quad (3.13)$$

Now define the three quantities:

$$K_{ij}^{-1} = \frac{1}{V} \int_{A_e} n_i m_j dA, \quad (3.14)$$

$$L_{ijk} = \frac{2}{V} \int_{A_e} n_i \frac{\partial q_j}{\partial x_k} dA, \quad (3.15)$$

$$N_{ij} = \frac{1}{V} \int_{A_e} n_i r_j dA \quad (3.16)$$

where K_{ij}^{-1} is the inverse of the permeability tensor as defined by Whitaker (1970). Equation 3.13 can then be written as

$$\frac{\partial \langle P \rangle}{\partial x_i} = -\mu K_{ij}^{-1} \langle v_j \rangle + \mu L_{ijk} \frac{\partial \langle v_j \rangle}{\partial x_k} + \mu N_{ij} \nabla^2 \langle v_j \rangle. \quad (3.17)$$

The permeability tensor K_{ij} itself (i.e., the inverse of K_{ij}^{-1} where $K_{ij} K_{ij}^{-1} = \delta_{ij}$, the Kronecker delta) must exist if the physical assumption is made that the average velocity is zero if and only if the average pressure is constant and the first and second derivatives of $\langle v_j \rangle$ are zero. Operating on Eq. 3.17 with K_{ij} and solving for $\langle v_i \rangle$:

$$\langle v_i \rangle = -\frac{K_{ij}}{\mu} \frac{\partial \langle P \rangle}{\partial x_j} + K_{ij} L_{jmk} \frac{\partial \langle v_m \rangle}{\partial x_k} + K_{ij} N_{jm} \nabla^2 \langle v_m \rangle \quad (3.18)$$

where $\langle v_i(x) \rangle$ and $\langle P(x) \rangle$ are the average velocity and pressure in the boundary region between the fluid and porous medium. The first term on the righthand side of Eq. 3.18 represents a Darcy flow and the third term is obviously related to the viscous term in the Navier-Stokes equation.

ORDER OF MAGNITUDE CONSIDERATIONS

Based on order of magnitude scaling arguments, it can be shown that Eq. 3.18 reduces to the Darcy equation and to the Navier-Stokes creep flow equation for points entirely within the porous medium below the boundary layer and within the fluid above the layer respectively. Inside the boundary layer, the three terms on the right side of Eq. 3.18 are of the same order and each must be retained.

Point x within Fluid Region $y \geq L$

In this region, the average quantities approximately equal the point quantities so that $v = \langle v \rangle$ and $\langle P \rangle = P$ which implies that the map $M_{ij} = \delta_{ij}$ so that $\partial M_{ij} / \partial x_k = \nabla^2 M_{ij} = 0$ and $m_j = \partial q_j / \partial x_k = 0$. From the definitions of K_{ij}^{-1} and L_{ijk} given by Eqs. 3.14 and 3.15, we have $K_{ij}^{-1} = L_{ijk} = 0$ (i.e., infinite permeability). The proper equation to use in this case is Eq. 3.17 which becomes

$$\frac{\partial P}{\partial x_i} = \mu N_{ij} \nabla^2 v_j. \quad (4.1)$$

To show that Eq. 4.1 reduces to the Navier-Stokes creep flow

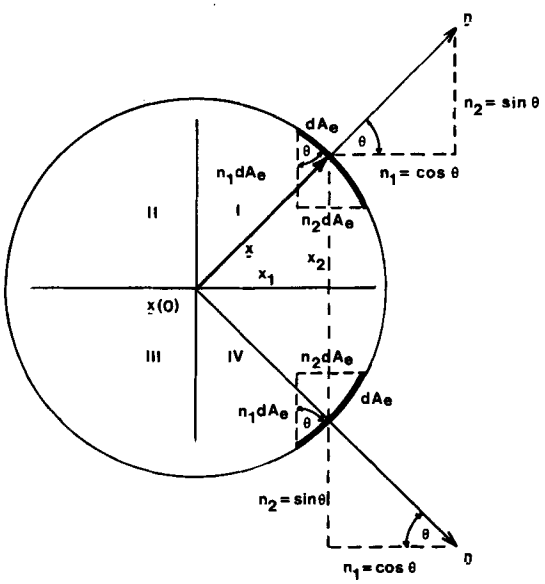

Figure 6. The averaging volume is considered to be a circular cylinder of unit depth for simplicity of calculation. The origin is at $x(0) = 0$ and all points within V lie entirely in the fluid region $y \geq L$ so that $A_i = 0$ and A_e is the total surface area. Two mirror points $x(s_b) \epsilon A_e$ are shown. The components of n and x are all positive in quadrant I and negative in III, while n_1, x_2 and n_2, x_1 have opposite signs in quadrants II and IV.

equation, we examine N_{ij} for the cylindrical volume of Figure 6. From Eqs. 3.8 and 3.16, it is clear that

$$r_j = \int_0^{s_b} e_{s_i} M_{ij} ds = \int_0^{s_b} e_{s_j} ds = \int_0^{s_b} \frac{dx_j ds}{d_s} = x_j(s_b) - x_j(0), \quad (4.2)$$

and since $x_j(0) = 0$ in this calculation,

$$N_{ij} = \frac{1}{V} \int_{A_e} n_i x_j dA \quad (4.3)$$

where A_e is the entire surface area in the fluid region.

From Figure 6, we note that

$$dV = (n_1 dA_e)x_1 \text{ and } dV = (n_2 dA_e)x_2$$

so that the total volume is

$$V = \int_{A_e} n_1 x_1 dA = \int_{A_e} n_2 x_2 dA$$

since $n_1 x_1$ and $n_2 x_2$ are positive in all four quadrants. The products $n_2 x_1$ and $n_1 x_2$ cancel out in I, IV and II, III so that

$$\int_{A_e} n_1 x_2 dA \text{ and } \int_{A_e} n_2 x_1 dA = 0$$

which gives

$$\delta_{ij} = \frac{1}{V} \int_{A_e} n_i x_j dA. \quad (4.4)$$

Comparing Eqs. 4.3 and 4.4, we have $N_{ij} = \delta_{ij}$ and Eq. 4.1 reduces to $\partial P/\partial x_i = \mu \nabla^2 v_i$.

Point x within Porous Medium $y \leq -L$

Point quantities in this region are scaled with the microscopic length d whereas averaged quantities are scaled with the macroscopic process length L_p (Figure 4). Restricting to the condition $L_p \gg L$ implies that all approximations made previously in the development of this theory (i.e., $l \ll L$ and $d \ll l$) are still valid. If $\hat{U}$ is the characteristic velocity within the porous medium, $\nabla v = 0(\hat{U}/d)$ and $\nabla \langle v \rangle = 0(\hat{U}/L_p)$. Since $v_i = M_{ij} \langle v_j \rangle$ and both the point and average velocities are of $0(\hat{U})$, it is clear that $M_{ij} = 0(1)$. (Note that this is also obvious from $\langle M_{ij} \rangle = \delta_{ij}$.) Derivatives of M_{ij} are thus $\partial M_{ij}/\partial x_k = 0(1/d)$ and $\nabla^2 M_{ij} = 0(1/d^2)$.

Consider now Eqs. 3.6, 3.7 and 3.8 which define m_j, $\partial q_j/\partial x_k$ and r_j in terms of integrals over a curve of length $s_b = 0(l)$. Scaling these equations gives

$$m_j = 0\left(\frac{l}{d^2}\right), \quad \frac{\partial q_j}{\partial x_k} = 0\left(\frac{l}{d}\right)$$

and $r_j = 0(l)$. If these orders of magnitude are used in Eqs. 3.14, 3.15 and 3.16, we have

$$K_{ij}^{-1} = 0\left[(m_j)\left(\frac{A}{V}\right)\right] = 0\left[\left(\frac{l}{d^2}\right)\left(\frac{1}{l}\right)\right] = 0(d)^{-2}, \quad (4.5)$$

$$L_{ijk} = 0\left[\left(\frac{\partial q_j}{\partial x_k}\right)\left(\frac{A}{V}\right)\right] = 0\left[\left(\frac{l}{d}\right)\left(\frac{1}{l}\right)\right] = 0(d),^{-1}, \quad (4.6)$$

and

$$N_{ij} = 0\left[(r_j)\left(\frac{A}{V}\right)\right] = 0\left[(l)\left(\frac{1}{l}\right)\right] = 0(1). \quad (4.7)$$

(Note that the permeability tensor K_{ij} is $0(d^2)$ as expected.)

The order of magnitude of the three terms on the right side of Eq. 3.17 are thus respectively

$$\frac{\partial \langle P \rangle}{\partial x_i} = 0\left(\mu \frac{\hat{U}}{d^2}\right) + 0\left(\frac{\mu \hat{U}}{dL_p}\right) + 0\left(\frac{\mu \hat{U}}{L_p{}^2}\right)$$

so that for $d \ll L_p$, we have

$$\frac{\partial \langle P \rangle}{\partial x_i} = -\mu K_{ij}^{-1} \langle v_j \rangle$$

which can be inverted to give the usual Darcy equation in the porous region:

$$\langle v_i \rangle = -\frac{K_{ij}}{\mu} \frac{\partial \langle P \rangle}{\partial x_j}. \quad (4.8)$$

Point x Inside Boundary Layer Region $-L \leq y \leq L$

Suppose that x is near the mean surface and that the gradients of point and average quantities are scaled with the same length L so that Eq. 1.5 is applicable. For this case, we find that $m_j = 0(l/L^2)$, $\partial q_j/\partial x_k = 0(l/L)$ and $r_j = 0(l)$ so that $K_{ij}^{-1} = 0(L^{-2})$, $L_{ijk} = 0(L^{-1})$ and $N_{ij} = 0(1)$. Substituting into Eq. 3.17 gives

$$\frac{\partial \langle P \rangle}{\partial x_i} = 0\left(\frac{\mu U}{L^2}\right)$$

for each of the terms.

Clearly, since the three terms are of the same order of magnitude, each should be retained in the slip condition at the porous fluid interface and so all of Eq. 3.18 is necessary for a consistent boundary condition.

RESULT FOR ISOTROPIC MEDIA

The boundary layer region is assumed to be isotropic even though the porosity changes rapidly there and so the material properties are independent of the orientation of the coordinate axes. If Eq. 3.18 is regarded as a pseudoconstitutive equation for $\langle v_i \rangle$, isotropy implies that the tensor quantities in the equation remain unchanged under an orthogonal transformation of coordinates.

Referring to Eqs. 3.6 through 3.8 and 3.14 through 3.16, it is noted that they are all of the same form and differ principally in the order of derivatives of M_{ij}. Since K_{ij}^{-1} and N_{ij} must be second-order isotropic tensors, they can be written in the form (Fung, 1977):

$$K_{ij}^{-1} = K^{-1}(x)\delta_{ij} \text{ or } K_{ij} = K(x)\delta_{ij}, \quad (5.1)$$

and

$$N_{ij} = N(x)\delta_{ij}. \quad (5.2)$$

This means that the map M_{ij} must be such that material isotropy is produced with respect to indices i and j in Eqs. 3.14 through 3.16. Tensor L_{ijk} which is of third order should then be regarded as isotropic in indices i and j, but not k, to be consistent with the other two equations. Thus, we can express L_{ijk} as

$$L_{ijk} = L_k(x)\delta_{ij}. \quad (5.3)$$

Substitution of Eqs. 5.1, 5.2, and 5.3 into Eq. 3.18 gives the slip condition valid in the isotropic boundary layer region:

$$\langle v_i \rangle = -\frac{K}{\mu} \frac{\partial \langle P \rangle}{\partial x_i} + KL_k \frac{\partial \langle v_i \rangle}{\partial x_k} + KN \nabla^2 \langle v_i \rangle \quad (5.4)$$

where $K(x)$ is the permeability of the medium.

As an example, consider the rectilinear motion $\langle v \rangle = \langle v_1(y) \rangle e_1$ shown in Figure 4. Equation 5.4 reduces to

$$\langle v_1 \rangle = -\frac{K}{\mu} \frac{d\langle P \rangle}{dx} + KL_2 \frac{d\langle v_1 \rangle}{dy} + KN \frac{d^2 \langle v_1 \rangle}{dy^2} \quad (5.5)$$

at the mean surface $y = 0$.

Comparing Eq. 5.5 to Beavers and Joseph's result given in the Scope, we see that the first two terms are the same except that the first term represents the Darcy flow at the mean surface $y = 0$ in Eq. 5.5 and within the porous medium $y = -L$ in Beavers and Joseph's equation.

Equating the second terms will provide us with a mathematical definition of the dimensionless parameter α:

$$\alpha = \frac{K^{-\frac{1}{2}}}{L_2}. \quad (5.6)$$

For this flow, $\langle v_j \rangle = \langle v_1 \rangle$ so that $v_i = M_{i1}\langle v_1 \rangle$ and Eqs. 3.15 and 3.7 give:

$$\frac{\partial q_1}{\partial x_2} = \int_0^{sb} e_{s_i} \frac{\partial M_{i1}}{\partial x_2} ds = \int_{x_0}^{x(s_b)} \frac{\partial M_{i1}}{\partial y} dx_i, \tag{5.7}$$

and

$$L_{112} = \frac{2}{V} \int_{Ae} n_1 \frac{\partial q_1}{\partial x_2} dA = L_2 \delta_{11}. \tag{5.8}$$

Solving for L_2:

$$L_2 = \frac{2}{V} \int_{A_e} \left\{ \int_{x_0}^{x(s_b)} \frac{\partial M_{i1}}{\partial y} dx_i \right\} dA. \tag{5.9}$$

Similarly Eqs. 3.6 and 3.14 yield K^{-1}:

$$m_1 = - \int_{x_0}^{x(s_b)} \nabla^2 M_{i1} dx_i, \tag{5.10}$$

and

$$K_{11}^{-1} = \frac{1}{V} \int_{A_e} n_1 m_1 dA = K^{-1} \delta_{11}.$$

Solving for K^{-1}:

$$K^{-1} = - \frac{1}{V} \int_{A_e} n_1 \left\{ \int_{x_0}^{x(s_b)} \nabla^2 M_{i1} dx_i \right\} dA. \tag{5.11}$$

Equation 5.6 now reads:

$$\alpha = \frac{\left[- \frac{1}{V} \int_{A_e} n_1 \left\{ \int_{x_0}^{x(s_b)} \nabla^2 M_{i1} dx_i \right\} dA \right]^{1/2}}{\frac{2}{V} \int_{A_e} n_1 \left\{ \int_{x_0}^{x(s_b)} \frac{\partial M_{i1}}{\partial y} dx_i \right\} dA.} \tag{5.12}$$

DISCUSSION

The theory presented in this paper has been based on the basic Navier-Stokes equations of motion for steady, incompressible, negligible inertia flow with constant viscosity. The model is predicated on the existence of four characteristic length scales which are ordered according to $L_p \gg L \gg l \gg d$ where L_p is the macroscopic process length, L is the boundary layer length, l is the averaging volume scale, and d is the microscopic length scale related to the grain or pore size. In addition, there is assumed to exist a linear transformation $M_{ij}(x)$ for all x from the average velocity to the point velocity (Eq. 2.1). This is a strong assumption; however, M_{ij} can be shown to exist in the interior of the porous medium and in the totally fluid space so that it appears reasonable to postulate its existence within the boundary layer as well.

The resulting slip boundary condition (Eq. 3.18) which is valid at the mean surface $y = 0$ gives the average velocity $\langle v \rangle$ in terms of the gradients of $\langle P \rangle$ *and* $\langle v \rangle$ for a general anisotropic medium. There are three terms present which are of the same order according to the scaling arguments used in the boundary region $-L \leq y \leq L$. The first two terms are of the same form as Beavers et al. (1967, 1970, 1974) empirical slip boundary condition which in this theory consists of a Darcy flow term, $-(K_{ij}/\mu)\partial\langle P \rangle/\partial x_j$, and a first-order derivative term in average velocity, $K_{ij}L_{jmk}\partial\langle v_m \rangle/\partial x_k$. In addition to these two terms, there is another term of second order in derivatives of average velocity, $K_{ij}N_{jm}\nabla^2\langle v_m \rangle$, which must be included for the slip condition to be valid.

If the complete Eq. 3.18 is examined via an order of magnitude scaling analysis for points below the boundary region, the first term is dominant and Darcy's equation results; likewise, for a point in the entirely fluid region, the third term is dominant and the Navier-Stokes equation results. At the mean surface $y = 0$, however, no one term dominates and all three are necessary.

For isotropic media, the second terms of Eqs. 3.18 and Beavers equation can be compared to determine the previously undefined dimensionless parameter α (Eq. 5.12) in terms of a line integral over an arbitrary curve lying within the averaging volume V, a surface integral over the entrance and exit fluid areas A_e, and the transformation M_{ij}. Since M_{ij} is a function of the structure of the medium alone, α is also a function of the structure and should be independent of the Newtonian fluid as long as the creep flow Navier-Stokes equations describe the motion in the fluid space V_f.

APPENDIX

Consider an integral of the form

$$\int_V f \nabla^n \langle \psi \rangle dV$$

where f and ψ are tensor fields and $n = 0,1,2, \ldots$ Using Taylor series expansions for f and $\nabla^n\langle \psi \rangle$ it can be shown that

$$f\nabla^n\langle \psi \rangle dV = \left(\int_V f dV \right) \nabla^n \langle \psi \rangle + 0\left[\left(\frac{l}{L} \right) \langle f \rangle \nabla^n \langle \psi \rangle \right] \tag{A.1}$$

so that $\nabla^n\langle \psi \rangle$ is essentially removed from the integral. For example, for $n = 0$ and $l \ll L$, the equation above is

$$\int_V f\langle \psi \rangle dV = \left(\int_V f dV \right) \langle \psi \rangle$$

or

$$\langle f\langle \psi \rangle \rangle = \langle f \rangle \langle \psi \rangle. \tag{A.2}$$

The volume average of $f\nabla^n\langle \psi \rangle$ at the point x is given by

$$\langle f(x)\nabla^n\langle \psi(x) \rangle \rangle = \frac{1}{V} \int_V f(r)\nabla^n\langle \psi(r) \rangle dV$$

where x is regarded as fixed and h varies over the volume so that $f(r)\nabla^n\langle \psi(r) \rangle$ is the value of the integrand at $r = x + h$ (Figure 2).

Expanding $\nabla^n\langle \psi(r) \rangle$ in a Taylor series about x, we have

$$\nabla^n\langle \psi(r) \rangle = \nabla^n\langle \psi(x) \rangle + h_i \frac{\partial \nabla^n}{\partial x_i} \langle \psi(x) \rangle + \frac{h_i h_j}{2!} \frac{\partial^2 \nabla^n \langle \psi(x) \rangle}{\partial x_i \partial x_j} + \ldots$$

Substituting the expansion above into the integral gives:

$$\langle f(x)\nabla^n\langle \psi(x) \rangle \rangle = \frac{1}{V} \int_V f(r) \left[\nabla^n \langle \psi(x) \rangle + h_i \frac{\partial}{\partial x_i} \nabla^n \langle \psi(x) \rangle + \frac{h_i h_j}{2!} \frac{\partial^2 \nabla^n}{\partial x_i \partial x_j} \langle \psi(x) \rangle + \ldots \right] dV$$

Since x is fixed, derivatives of $\nabla^n\langle \psi \rangle$ can be removed from the integral and

$$\langle f(x)\nabla^n\langle \psi(x) \rangle \rangle = \left(\frac{1}{V} \int_V f(r) dV \right) \nabla^n \langle \psi(x) \rangle + \left(\frac{1}{V} \int_V f h_i dV \right) \frac{\partial}{\partial x_i} \nabla^n \langle \psi(x) \rangle + \ldots$$

Now $h = 0(l)$, but derivatives of the averaged quantities are scaled with the boundary layer length L so $\partial/\partial x_i \nabla^n\langle \psi \rangle = 0(\nabla^n\langle \psi \rangle/L)$ and

$$\langle f(x)\nabla^n\langle \psi(x) \rangle \rangle = \langle f(x) \rangle \nabla^n \langle \psi(x) \rangle + 0\left[(l\langle f \rangle) \left(\frac{\nabla^n \langle \psi \rangle}{L} \right) \right] + 0\left[\frac{l^2}{L^2} \langle f \rangle \nabla^n \langle \psi \rangle \right]$$

$$= \langle f(x) \rangle \nabla^n \langle \psi(x) \rangle + 0\left[\left(\frac{l}{L} \right) \langle f \rangle \nabla^n \langle \psi \rangle \right] + \ldots$$

Restricting the length scales to be such that $l \ll L$, we have the result

$$\langle f(x)\nabla^n\langle \psi(x) \rangle \rangle = \langle f(x) \rangle \nabla^n \langle \psi(x) \rangle$$

or

$$\int_V f\nabla^n\langle \psi \rangle dV = \left(\int_V f dV \right) \nabla^n \langle \psi \rangle. \tag{A.3}$$

Similar results can be found for area and line integrals using the same approach as long as $l \ll L$. In particular, it is obvious that

$$\int_A f\nabla^n\langle\psi\rangle dA = \left(\int_A f dA\right)\nabla^n\langle\psi\rangle \tag{A.4}$$

and

$$\int_0^{s_b} f\nabla^n\langle\psi\rangle ds = \left(\int_0^{s_b} f ds\right)\nabla^n\langle\psi\rangle \tag{A.5}$$

if $A = 0(l^2)$ and $S_b = 0(l)$.

NOTATION

A_e = entrance and exit surface areas of volume V_f
A_i = interior surfaces of solid volume V_s inside V
d = microscopic length scale
e_s = unit tangent to curve s
e_1, e_2 = unit vectors in x and y directions, respectively
K = permeability
K_{ij} = permeability tensor
l = length scale of averaging volume V
L = length scale of boundary region
L_p = length scale of macroscopic process
L_{ijk} = tensor defined in text via an integral over A_e
M_{ij} = a linear transformation that maps $\langle v\rangle$ into v
n = unit outer normal
N_{ij} = tensor defined in text via an integral over A_e
m_j, r_j, & $\frac{\partial q_j}{\partial x_k}$ = vectors and tensors defined in text via line integrals
$P, \langle P\rangle$ = point and average pressure
s = arc length along curve going from 0 to s_b
$U, \hat{U}$ = characteristic velocity of fluid above and below boundary region
$v, \langle v\rangle$ = point and average velocity
V = averaging volume
V_f = total fluid volume within V
V_s = total solid volume within V
x = point in space

Greek Letters

α = dimensionless quantity that depends on the structure of the porous medium
β = void volume distribution function
δ_{ij} = Kronecker delta
ζ = equation of surface above the xz plane
Φ = porosity
ψ = a continuous tensor field
μ = fluid viscosity (Newtonian)

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Effective Diffusion and Conduction in Two-Phase Media: a Unified Approach

A multiscale analysis based on two widely different length scales that exist in stagnant two-phase media is carried out to investigate diffusion and conduction in periodic and random systems. A variational form of the resulting unit cell equations allows one to demonstrate its equivalence to the cell equations of other independent approaches, thus unifying these widely different methods. The variational approach is also shown to facilitate numerical studies and to provide a link between random and periodic media by yielding an upper bound for the effective diffusivity in random media in terms of the periodic one.

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SCOPE

The need to describe transport phenomena in multiphase media with complex geometries in terms of simple "homogenized" equations with "effective" or "bulk" transport parameters have inspired a surfeit of widely different approaches to the problem for both periodic and random media. These approaches yield seemingly different methods for deriving the effective diffusivity or conductivity tensors. Perhaps because of the apparent divergence of these approaches, numerical studies and comparisons of theory with experimental data have been lacking. This paper aims to unify various approaches for periodic media and establish a relationship between the periodic and random model for heterogeneous media. Numerical studies are also performed with the most efficient numerical schemes and the results are compared to experimental data.